

Ram

$$E^2 = p^2 c^2 + m^2 c^4$$

$$E \rightarrow i\hbar \frac{\partial}{\partial t}$$

$$p \rightarrow -i\hbar \nabla$$

$$-\hbar^2 \frac{\partial^2 \phi}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \phi + m^2 c^4 \phi$$

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2} \right) \phi = 0$$

$$\phi = A e^{i(p \cdot x - E t) / \hbar}$$

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{-E^2}{\hbar^2} \phi$$

$$\nabla^2 \phi = \frac{-p^2}{\hbar^2} \phi$$

$$\cancel{-E^2} - \frac{E^2}{c^2} + p^2 + m^2 c^2 = 0$$

$$E^2 = p^2 c^2 + m^2 c^4$$

* Now Dirac wanted something like:

$$i\hbar \partial_t \psi = H \psi$$

$$H = c \alpha \cdot p + \beta mc^2$$

s.t. $H^2 = (p^2 c^2 + m^2 c^4) I$

That only works if:

$$\alpha_i^2 = \beta^2 = I \quad \alpha_i \alpha_j + \alpha_j \alpha_i = 0 \quad (i \neq j)$$
$$\alpha_i \beta + \beta \alpha_i = 0$$

- Matrices can anticommute
and the smallest matrices that satisfy all these
requirements are 4×4

$$i\hbar \partial_t \psi = (-i\hbar c \alpha \cdot \nabla + \beta mc^2) \psi$$

$$i\hbar \left(\frac{\partial}{\partial t} + \nabla \alpha - \beta mc \right) \psi = 0$$

$$(i\hbar \gamma^\mu \partial_\mu - mc) \psi = 0 \quad \gamma^0 = \beta, \quad \gamma^i = \beta \alpha_i, \quad \partial_0 = \frac{1}{c} \partial_t$$

A Fast Free Electron

$$p = \sqrt{3} mc \quad e^{i(pz - Et)/\hbar}$$

$$\psi = \psi(z, t) = \begin{pmatrix} \sqrt{3}/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} e^{i(pz - Et)/\hbar}$$

Electron moving in $\hat{x} + \hat{z}$ with spin up

$$i\hbar \frac{\partial \psi}{\partial t} = (-i\hbar c \alpha_z \frac{\partial}{\partial z} + \beta mc^2) \psi$$

$$\alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\psi(z, t) = \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} e^{i(pz - Et)/\hbar}$$

$$E \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} = i\hbar \frac{1(E)}{\hbar} \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} = -i\hbar \frac{(ip)}{\hbar} \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} \beta + mc^2 \beta \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix}$$

$$E \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} = (cp \alpha_z + mc^2 \beta) \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix}$$

$$E \alpha_z$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ a \\ 0 \end{pmatrix}$$

$$c \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} \cdot \begin{pmatrix} b \\ 0 \\ a \\ 0 \end{pmatrix} = c(ab + ab) = 0$$

$$= 2cab - \frac{\sqrt{3}}{2}c$$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ -b \\ 0 \end{pmatrix}$$

$$Ea = cpb + mc^2a$$

$$Eb = cpa - mc^2b$$

$$(E - mc^2)a = cpb \quad (1)$$

$$(E + mc^2)b = cpa \quad (2)$$

$$(E - mc^2)a = \frac{c^2 p^2}{E + mc^2} a$$

$$E^2 - m^2 c^4 = c^2 p^2 \Rightarrow E^2 = p^2 c^2 + m^2 c^4$$

$$p = \sqrt{3} mc$$

$$E^2 = 4 m^2 c^4 \Rightarrow E = 2mc^2$$

$$b = \frac{cp}{E + mc^2} a = \frac{\sqrt{3} mc^2}{2mc^2} a = \frac{a}{\sqrt{3}}$$

$$a^2 + b^2 = 1$$

$$a^2 + \frac{a^2}{3} = 1 \Rightarrow 4a^2 = 3 \Rightarrow a = \frac{\sqrt{3}}{2} \quad b = \frac{1}{2}$$

$$\psi(z, t) = \begin{pmatrix} \sqrt{3}/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} e^{i(\sqrt{3}mcz - 2mc^2t)/\hbar}$$