

Ram

$$E^2 = p^2 c^2 + m^2 c^4$$

$$E \rightarrow i\hbar \frac{\partial}{\partial t} \quad p \rightarrow -i\hbar \nabla$$

$$-\hbar^2 \frac{\partial^2 \phi}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \phi + m^2 c^4 \phi$$

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2} \right) \phi = 0$$

$$\phi = A e^{i(p \cdot x - E t) / \hbar}$$

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{-E^2}{\hbar^2} \phi$$

$$\nabla^2 \phi = \frac{-p^2}{\hbar^2} \phi$$

$$-\cancel{E^2} - \frac{E^2}{c^2} + p^2 + m^2 c^2 = 0$$

$$E^2 = p^2 c^2 + m^2 c^4$$

* Now Dirac wanted something like:

$$i\hbar \partial_t \psi = H \psi$$

$$H = c \alpha \cdot p + \beta mc^2$$

s.t. $H^2 = (p^2 c^2 + m^2 c^4) I$

That only works if:

$$\alpha_i^2 = \beta^2 = I \quad \alpha_i \alpha_j + \alpha_j \alpha_i = 0 \quad (i \neq j)$$
$$\alpha_i \beta + \beta \alpha_i = 0$$

- Matrices can anticommute
and the smallest matrices that satisfy all these
requirements are 4×4

$$i\hbar \partial_t \psi = (-i\hbar c \alpha \cdot \nabla + \beta mc^2) \psi$$

$$i\hbar \left(\frac{\partial}{\partial t} + \nabla \alpha - \beta mc \right) \psi = 0$$

$$(i\hbar \gamma^\mu \partial_\mu - mc) \psi = 0 \quad \gamma^0 = \beta, \gamma^i = \beta \alpha_i, \partial_0 = \frac{1}{c} \partial_t$$

A Fast Free Electron

$$p = \sqrt{3} mc \quad e^{i(pz - Et)/\hbar}$$

$$\psi = \psi(z, t) = \begin{pmatrix} \sqrt{3}/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} e^{i(pz - Et)/\hbar}$$

Electron moving in $\hat{x} + \hat{z}$ with spin up

$$i\hbar \frac{\partial \psi}{\partial t} = (-i\hbar c \alpha_z \frac{\partial}{\partial z} + \beta mc^2) \psi$$

$$\alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\psi(z, t) = \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} e^{i(pz - Et)/\hbar}$$

$$E \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} = i\hbar \frac{1(-E)}{\hbar} \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} = -i\hbar \frac{(ip)}{\hbar} \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} \beta + mc^2 \beta \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix}$$

$$E \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} = (cp \alpha_z + mc^2 \beta) \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix}$$

$$E \alpha_z =$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ a \\ 0 \end{pmatrix}$$

$$c \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} \cdot \begin{pmatrix} b \\ 0 \\ a \\ 0 \end{pmatrix} = c(ab + ab) = 0$$

$$= 2cab - \frac{\sqrt{3}}{2}c$$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ -b \\ 0 \end{pmatrix}$$

$$Ea = cpb + mc^2a$$

$$Eb = cpa - mc^2b$$

$$(E - mc^2)a = cpb \quad (1)$$

$$(E + mc^2)b = cpa \quad (2)$$

$$(E - mc^2)a = \frac{c^2 p^2}{E + mc^2} a$$

$$E^2 - m^2 c^4 = c^2 p^2 \Rightarrow E^2 = p^2 c^2 + m^2 c^4$$

$$p = \sqrt{3} mc$$

$$E^2 = 4 m^2 c^4 \Rightarrow E = 2mc^2$$

$$b = \frac{cp}{E + mc^2} a = \frac{\sqrt{3} mc^2}{2mc^2} a = \frac{a}{\sqrt{3}}$$

$$a^2 + b^2 = 1$$

$$a^2 + \frac{a^2}{3} = 1 \Rightarrow 4a^2 = 3 \Rightarrow a = \frac{\sqrt{3}}{2} \quad b = \frac{1}{2}$$

$$\psi(z, t) = \begin{pmatrix} \sqrt{3}/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} e^{i(\sqrt{3}mcz - 2mc^2t)/\hbar}$$

APPENDIX A / CORRECTIONS

Making the derivation precise

The first four pages are Nimeesh S K's original handwritten notes, kept intact. These three typed pages are an editorial supplement prepared with AI assistance: corrections, interpretation and a bridge to the next topic. Page references below refer to the original scans.

1. Page 2: keep the mass term outside $i\hbar$

Starting from the Hamiltonian, move every term to the left and divide by c :

$$\begin{aligned} i\hbar\partial_t\psi &= (-i\hbar c\boldsymbol{\alpha}\cdot\nabla + \beta mc^2)\psi, \\ \left[i\hbar\left(\frac{1}{c}\partial_t + \boldsymbol{\alpha}\cdot\nabla\right) - \beta mc \right] \psi &= 0. \end{aligned}$$

The intermediate handwritten line places the mass term inside the $i\hbar$ factor. Use the second line above instead. Now multiply *from the left* by β and use $\beta^2 = I$:

$$\boxed{(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0}, \quad \gamma^0 = \beta, \quad \gamma^i = \beta\alpha_i.$$

The final covariant equation in the scan is correct. The missing step is the left multiplication by β ; α_i and γ^i are different matrices.

2. Fix the conventions before manipulating indices

Here $x^\mu = (ct, x, y, z)$, $\partial_\mu = (c^{-1}\partial_t, \partial_x, \partial_y, \partial_z)$ and $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Repeated spacetime indices run from 0 to 3; spatial indices run from 1 to 3. Thus

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I, \quad (\gamma^0)^2 = I, \quad (\gamma^i)^2 = -I.$$

The braces mean $AB + BA$. The matrices anticommute for *different* indices; their squares are not zero. The minimal complex Dirac matrices in $3 + 1$ dimensions are 4×4 . [1]

3. Pages 3–4: use one momentum consistently

The opening momentum on page 3 appears to read $\sqrt{2}mc$, but page 4 and the displayed spinor use $\boxed{p = \sqrt{3}mc}$. Taking the positive-energy branch gives

$$E = 2mc^2, \quad \frac{b}{a} = \frac{cp}{E + mc^2} = \frac{1}{\sqrt{3}}, \quad a = \frac{\sqrt{3}}{2}, \quad b = \frac{1}{2}.$$

These coefficients are correct for $p = \sqrt{3}mc$. If $p = \sqrt{2}mc$ was intended, then $E = \sqrt{3}mc^2$ and the coefficients must change. For general positive p , with real a, b and $a^2 + b^2 = 1$,

$$a = \sqrt{\frac{E + mc^2}{2E}}, \quad b = \sqrt{\frac{E - mc^2}{2E}}.$$

APPENDIX B / PHYSICAL MEANING

What the equations are saying

4. Klein–Gordon is still a valid equation

Page 1 correctly recovers $E^2 = p^2c^2 + m^2c^4$. Both energy signs are allowed. For a complex, massive KG wave, a conventional conserved density is

$$\rho_{\text{KG}} = \frac{i\hbar}{2mc^2} (\phi^* \partial_t \phi - \phi \partial_t \phi^*).$$

For $\phi = Ae^{i(\mathbf{p}\cdot\mathbf{x} - Et)/\hbar}$, direct substitution gives $\rho_{\text{KG}} = E|A|^2/(mc^2)$. Its sign can be negative, so it cannot serve as a general positive single-particle probability density. [2]

That does not invalidate KG. In field theory a complex scalar has a conserved charge current, with opposite charges for particles and antiparticles; a real scalar describes neutral quanta. A scalar KG field describes spin 0. Each component of a free Dirac solution also obeys KG, but the components are linked by the first-order Dirac equation. [1,3]

5. The Dirac current checks the electron example

For a first-quantized Dirac spinor,

$$\rho = \psi^\dagger \psi \geq 0, \quad \mathbf{j} = c\psi^\dagger \boldsymbol{\alpha} \psi, \quad \partial_t \rho + \nabla \cdot \mathbf{j} = 0.$$

Equivalently $j^\mu = c\bar{\psi}\gamma^\mu\psi$, where $\bar{\psi} = \psi^\dagger\gamma^0$ and $j^0 = c\rho$. The electric current additionally carries the particle's charge. [2]

For the page 4 spinor $u = (a, 0, b, 0)^\top$, real a, b give

$$u^\dagger \alpha_z u = 2ab, \quad \frac{j_z}{\rho} = 2cab = \frac{\sqrt{3}}{2}c = \frac{pc^2}{E}.$$

The probability flows along $+z$ at the expected subluminal speed. The condition $u^\dagger u = 1$ normalizes the *spinor*, not a plane wave throughout infinite space. In a periodic box of volume V , use $\psi = ue^{i(pz - Et)/\hbar}/\sqrt{V}$; a localized particle requires a wave packet.

6. Four components are not four particles

In the Dirac basis used in the scans, the upper and lower pairs are coupled parts of one spinor. A nonzero lower pair does not mean an electron has become partly a positron. Particle and antiparticle states are identified through the energy-mode structure and field quantization, not by simply naming individual entries. [1,4]

For $p \ll mc$, $b/a = cp/(E + mc^2) \simeq p/(2mc)$: the lower pair is small. At $p = \sqrt{3}mc$, it is significant. This is a useful way to see how the nonrelativistic approximation begins to fail.

APPENDIX C / THE NEXT LAYER

Covariance, chirality and fields

7. Lorentz transformations have two jobs

For $x' = \Lambda x$, a spinor transforms as $\psi'(x') = S(\Lambda)\psi(x)$. The coordinate transformation changes the argument; S mixes the spinor components. For proper orthochronous transformations, covariance follows from

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu.$$

This makes the transformed equation have the same form. The four spinor entries are not the four coordinates of spacetime. [1]

8. Massless does not mean photon

Define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $P_{L,R} = (I \mp \gamma^5)/2$. In a chiral basis, writing the two Weyl components as ψ_L, ψ_R , the free equations are

$$\begin{aligned} i\hbar\partial_t\psi_L &= +i\hbar c \boldsymbol{\sigma} \cdot \nabla \psi_L + mc^2\psi_R, \\ i\hbar\partial_t\psi_R &= -i\hbar c \boldsymbol{\sigma} \cdot \nabla \psi_R + mc^2\psi_L. \end{aligned}$$

The mass term couples the two chiralities. When $m = 0$, they evolve independently: an initially absent component stays absent under free evolution. These are spin-1/2 Weyl equations; photons are spin-1 excitations of the electromagnetic field. [1,5]

“Left” is not a direction of travel. Helicity measures spin relative to momentum; chirality labels the spinor’s transformation type. For positive-energy massless particle states, left chirality corresponds to negative helicity and right chirality to positive helicity. Massive particles need not have matching chirality and helicity. Antiparticle conventions require separate care. [5]

9. Why second quantization comes next

Positive Dirac probability density does not remove negative-energy solutions. Quantization reinterprets the mode expansion in terms of operators that destroy particles and create antiparticles, with positive excitation energies. A state lives in Fock space; the field is an operator acting on that state. [4]

Start with occupation numbers and the oscillator ladder operators, then bosonic commutators and fermionic anticommutators. The latter enforce at most one identical fermion per single-particle state. Second quantization also works for nonrelativistic systems with conserved total particle number; number-changing processes become possible when the interactions allow them. [3,4]

Sources for the added explanations

- [1] David Tong, *Quantum Field Theory*, section 4: The Dirac Equation.
- [2] MIT 8.322, *The Dirac Equation*, chapter 1, sections 1.2–1.3.
- [3] David Tong, *Quantum Field Theory*, section 2: Free Fields.
- [4] David Tong, *Quantum Field Theory*, section 5: Quantizing the Dirac Field.
- [5] MIT 8.811, *Particle Physics II*, lecture 4: chirality and helicity.

The worked-example checks above are direct algebra using the scanned equations.